The dynamics of microscale plates submerged in fluid

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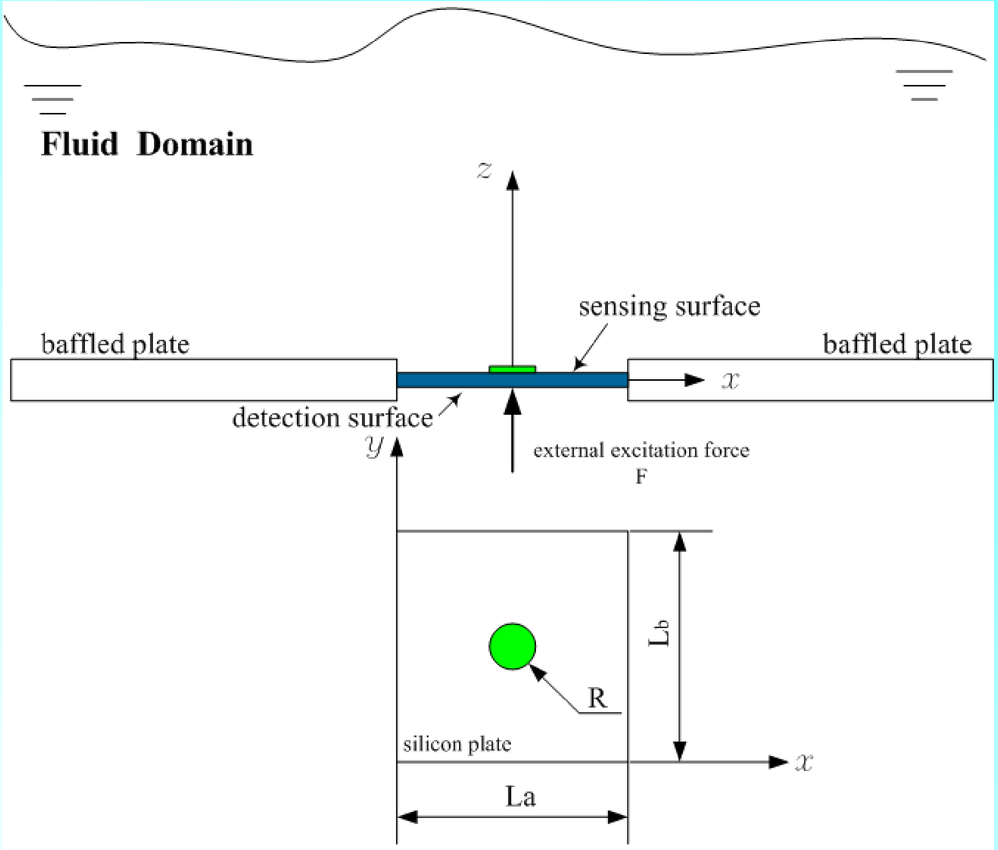
Dynamic Modelling

FIRST analytical solution with considerations of:

- **Distributed mass loading**
- **Fluid interaction**



$$D\nabla^4 w(x,y,t) + \rho_p h \frac{\partial^2 w(x,y,t)}{\partial t^2} + \overline{m}_c \mathcal{H}(x,y) \frac{\partial^2 w(x,y,t)}{\partial t^2} = p(x,y,t)$$

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Heaviside function:

$$\mathcal{H}(x,\ y) = \{H[x-(x_c-R)] - H[x-(x_c+R)]\} \cdot \left\{H\left[y-(y_c-\sqrt{R^2-x^2})\right] - H\left[y-(y_c+\sqrt{R^2-x^2})\right]\right\}$$

The classical thin plate theory is valid when:

$$\frac{\omega h}{\pi c_s} < 0.1$$

C_s is material shear wave speed

The displacement: a summation of a series of eigenfunctions:

$$w(x,\ y,\ t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} X_m(x) Y_n(y) \cdot \theta(t)$$

The effect of cross-modal coupling between fluid and plate:

1. Acoustic reactance force, generate inertial forces on the plate
2. Resistive force, appears as damping/energy dissipation due to acoustic radiation

Simplified Solution using Rayleigh-Ritz method

Assume fluid is inviscid (μ=0) and incompressible (c=∞)

$$\nabla^2 \Phi(x,\ y,z,t) = \frac{1}{c^2}\frac{\partial^2 \Phi(x,\ y,z,t)}{\partial t^2}$$

For steady state harmonic motion: $\Phi(x,\ y,\ z,\ t) = \phi(x,\ y,\ z)\mathrm{e}^{-\mathrm{i}\omega t}$.

$$\phi(x,\ y,z) = \frac{\mathrm{i}\omega}{2\pi}\iint_S \frac{w(\xi,\eta)\exp\left(\mathrm{i}k\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}\mathrm{d}\xi\,\mathrm{d}\eta,$$

The fluid-induced inertial effect is proportional to the kinetic energy of fluid:

$$T_\mathrm{f} = -\frac{1}{2}\rho_\mathrm{f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\phi(x,\ y,z)}{\partial z}\bigg|_{z=0}\phi(x,\ y,0)\ \mathrm{d}x\,\mathrm{d}y$$

$$T_\mathrm{f} = \frac{\rho_\mathrm{f}\omega^2}{4\pi}\iint_S\iint_S \frac{w(x,\ y)w(\xi,\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Solution using Rayleigh-Ritz method

Kinetic energy includes that of the plate, the fluid and the attached mass

$$\omega_{\mathrm{c}}^2 = \frac{U_{\mathrm{p}}}{T_{\mathrm{p}}^* + T_{\mathrm{f}}^* + T_{\mathrm{m}}^*}$$

$$U_{\mathrm{p}} = \frac{D}{2}\iint_S \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} \, \mathrm{d}x\,\mathrm{d}y$$

$$T_{\mathrm{p}}^* = \frac{1}{2}\rho_{\mathrm{p}} h \iint_S w^2(x,\, y)\ \mathrm{d}x\,\mathrm{d}y \qquad T_{\mathrm{f}}^* = T_{\mathrm{f}}/\omega^2$$

$$T_{\mathrm{m}}^* = \frac{1}{2}\iint_S \bar{m}_{\mathrm{c}} \mathcal{H}(x,\, y) w^2(x,\, y)\ \mathrm{d}x\,\mathrm{d}y.$$

The total energy: $V = U_{\mathrm{p}} - \omega^2\left(T_{\mathrm{p}}^* + T_{\mathrm{f}}^* + T_{\mathrm{m}}^*\right)$

Eigenfunctions can be obtained by minimizing V wrt unknown deflection coefficient:

$$\frac{\partial V}{\partial W_{qr}} = 0 \quad \Rightarrow \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{U_{\mathrm{p},mnqr} - \omega^2\left(T_{\mathrm{p},mnqr}^* + T_{\mathrm{f},mnqr}^* + T_{\mathrm{m},mnqr}^*\right)\right\} W_{mn} = 0,$$

Derive the eigenfunctions:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{U_{\mathrm{p},mnqr}-\omega^2(T^*_{\mathrm{p},mnqr}+T^*_{\mathrm{f},mnqr}+T^*_{\mathrm{m},mnqr})\}W_{mn}=0,$$

$$\{[\boldsymbol{A}]-\omega^2[\boldsymbol{I}]\}\{x\}=0,$$

$$\boldsymbol{A}=\boldsymbol{M}^{-1}\boldsymbol{K}$$

$$K_{ij}=U_{\mathrm{p},mnqr}$$
$$M_{ij}=T^*_{\mathrm{p},mnqr}+T^*_{\mathrm{f},mnqr}+T^*_{\mathrm{m},mnqr}$$
$$i=l(q-1)+r,\quad j=l(m-1)+n,\quad l\in\mathbb{N}^+$$

$$U_{\mathrm{p},mnqr}=\frac{D}{2}\iint_S\{\ddot{X}_m(x)\ddot{X}_q(x)Y_n(y)Y_r(y)+X_m(x)X_q(x)\ddot{Y}_n(y)\ddot{Y}_r(y)+2\nu\ddot{X}_m(x)X_q(x)Y_n(y)\ddot{Y}_r(y)+2(1-\nu)\dot{X}_m(x)\dot{X}_q(x)\dot{Y}_n(y)\dot{Y}_r(y)\}\mathrm{d}x\mathrm{d}y$$

$$T^*_{\mathrm{p},mnqr}=\frac{1}{2}\rho_{\mathrm{p}}h\iint_S X_m(x)X_q(x)Y_n(y)Y_r(y)\;\mathrm{d}x\,\mathrm{d}y$$

$$T^*_{\mathrm{f},mnqr}=\frac{\rho_{\mathrm{f}}}{4\pi}\iint_S\iint_S\frac{X_m(x)Y_n(y)X_q(\xi)Y_r(\eta)}{\sqrt{(x-\xi)^2+(y-\eta)^2}}\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Challenge: integral singularity in acoustic impedance

$$T^*_{\mathrm{m},mnqr}=\frac{1}{2}\bar{m}_{\mathrm{c}}\iint_S X_m(x)X_q(x)Y_n(y)Y_r(y)\mathcal{H}(x,\,y)\;\mathrm{d}x\,\mathrm{d}y.$$

Realistic model: fluid acoustic radiation and viscous effect

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho_p h \frac{\partial^2 w}{\partial t^2} = F(x,y,t)$$

$$F(x,y) = F_{ex}\delta(x-x_0)\delta(y-y_0) + F_{hydro}(x,y,0)$$

$$F_{hydro}(x,y,0,t) = F_{hydro}(x,y,0-,t) - F_{hydro}(x,y,0+,t)$$

$$F_{hydro}(x,y,0-,t) = -F_{hydro}(x,y,0+,t)$$

Under Non-slip condition, hydrodynamic force depends on the fluid stress tensor:

$$F_{hydro}(x,y,0+) = -\sigma_z + \frac{h}{2}\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y}\right)$$

Substitute fluid velocity into fluid stress tensors:

$$\sigma_z = -2\mu\nabla^2\phi + \rho_{f0}\dot{\phi} + 2\mu\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\psi_y}{\partial x\partial z} + \frac{\partial^2\psi_x}{\partial y\partial z}\right)$$

$$\tau_{zx} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi_z}{\partial y\partial z} - \frac{\partial^2\psi_x}{\partial y\partial x} - \frac{\partial^2\psi_y}{\partial z^2} + \frac{\partial^2\psi_y}{\partial x^2}\right)$$

$$\tau_{zy} = \mu\left(2\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\psi_y}{\partial x\partial y} - \frac{\partial^2\psi_z}{\partial x\partial z} - \frac{\partial^2\psi_x}{\partial y^2} + \frac{\partial^2\psi_x}{\partial z^2}\right)$$

The Navier-Stokes equation for the motion of a viscous compressible fluid:

$$\rho_{f0}\frac{\partial \mathbf{v}}{\partial t} - \mu\nabla^2\mathbf{v} - (\mu + \mu^v)\nabla(\nabla\cdot\mathbf{v}) + \nabla p = 0$$

The motion also satisfies the continuity equation: $\frac{\partial \rho_f}{\partial t} + \rho_{f0}\nabla\cdot\mathbf{v} = 0$

And momentum equation: $\frac{\partial p}{\partial \rho_f} = c^2$

The velocity field: $\mathbf{v} = \nabla\Phi + \nabla\times\Psi$ $\nabla\cdot\Psi = 0$

Consider harmonic motion:

$$\Phi(x,y,z,t) = \phi(x,y,z)e^{-i\omega t}$$

$$\Psi(x,y,z,t) = \psi(x,y,z)e^{-i\omega t}$$

$$\nabla^2\phi + k_l^2\phi = 0 \qquad k_l^2 = \frac{\omega^2/c^2}{1 - 4i\mu\omega/3\rho_{f0}c^2}$$

$$\nabla^2\psi + k_s^2\psi = 0 \qquad k_s^2 = \frac{i\rho_{f0}\omega}{\mu}$$

Apply double Fourier transforms:

$$\phi(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A\cdot\exp\left(ik_x x + ik_y y + \sqrt{k_l^2 - k_x^2 - k_y^2}\cdot z\right)dk_x dk_y$$

$$\psi(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} B\cdot\exp\left(ik_x x + ik_y y + i\sqrt{k_s^2 - k_x^2 - k_y^2}\cdot z\right)dk_x dk_y$$

Apply non-slip BCs at the fluid-plate interface:

$$\frac{\partial \mathbf{u}^p}{\partial t} = \mathbf{v}^f, \quad \vec{\boldsymbol{\sigma}}^p = \vec{\boldsymbol{\sigma}}^f$$

Expand it in the Cartesian coordinates,

$$v_x^f = v_x^p|_{z=0}, \quad v_y^f = v_y^p|_{z=0}, \quad v_z^f = v_z^p|_{z=0}$$

$$\begin{aligned} v_x^f &= \frac{\partial \phi}{\partial x} + \frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z} \\ v_y^f &= \frac{\partial \phi}{\partial y} + \frac{\partial \psi_x}{\partial z} - \frac{\partial \psi_z}{\partial x} \\ v_z^f &= \frac{\partial \phi}{\partial z} + \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} \end{aligned}$$

$$\begin{aligned} v_x^p &= -\frac{h}{2}\frac{\partial^2 w}{\partial x \partial t} \\ v_y^p &= -\frac{h}{2}\frac{\partial^2 w}{\partial y \partial t} \\ v_z^p &= \frac{\partial w}{\partial t} \end{aligned}$$

$$\nabla \cdot \Psi = 0$$

$$\begin{aligned} i\sqrt{k_l^2 - k_x^2 - k_y^2} \cdot A + ik_y B_x - ik_x B_y &= \tilde{\dot{w}} \\ k_x A + \sqrt{k_s^2 - k_x^2 - k_y^2} \cdot B_y + k_y B_z &= \frac{h}{2} k_x \tilde{\dot{w}} \\ -k_y A + \sqrt{k_s^2 - k_x^2 - k_y^2} \cdot B_x + k_x B_z &= \frac{h}{2} k_x \tilde{\dot{w}} \\ -k_x B_x - k_y B_y + \sqrt{k_s^2 - k_x^2 - k_y^2} \cdot B_z &= 0 \end{aligned}$$

$$\phi(x,y,z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} A \cdot \exp\left(ik_x x + ik_y y + \sqrt{k_l^2 - k_x^2 - k_y^2} \cdot z \right) dk_x dk_y$$

$$\psi(x,y,z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} B \cdot \exp\left(ik_x x + ik_y y + \ i\sqrt{k_s^2 - k_x^2 - k_y^2} \cdot z \right) dk_x dk_y$$

The analytical expression of the hydrodynamic force:

$$F_{hydro}(x,y,0) = 2 \cdot \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} T(k_x,k_y)\tilde{w} \exp(ik_x x + ik_y y) dk_x dk_y$$

The essential matrix form of the vibration:

$$\sum_{m}^{\infty} \sum_{n}^{\infty} (\mathbf{\Gamma} + i\omega \mathbf{I_{mnqr}}) \{W_{mn}\} = F_{qr} \quad q,r = 1,2,\cdots,\infty$$

Where external force:

$$F_{qr} = \iint_S F_{ex} \delta(x-x_0)\delta(y-y_0) X_q(x) Y_r(y) \, dx \, dy$$

Modal coefficient of the plate stiffness:

$$\mathbf{\Gamma} = M \left[\omega_{mn}^2 - \omega^2\right] \mathbf{E}$$

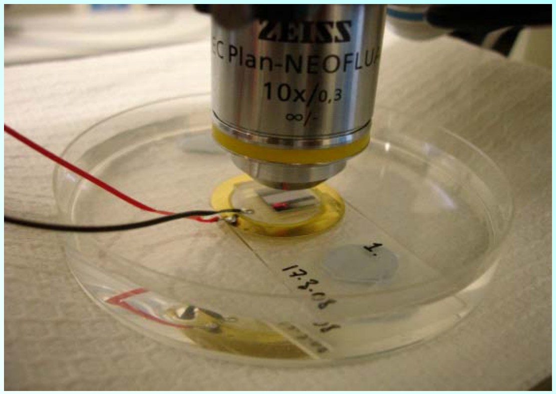
$$\omega_{mn}^2 = \frac{D}{\rho_p h} \left(k_m^4 + 2 \frac{\iint_S X_m \ddot{X}_m Y_n \ddot{Y}_n dx dy}{\iint_S X_m^2 Y_n^2 dx dy} + k_n^4 \right)$$

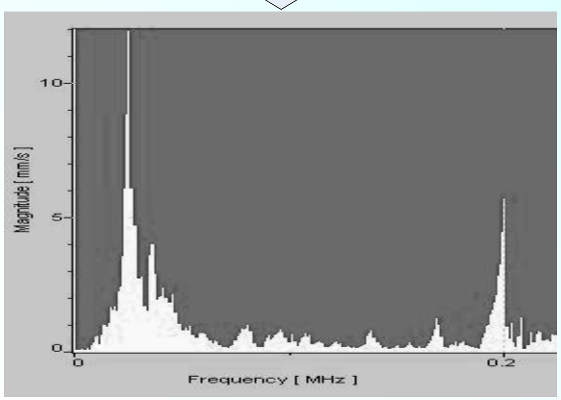
The direct and mutual mode fluid-loading impedance

$$I_{mnqr} = 2 \cdot \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} T(k_x,k_y) \chi_{mn}(k_x,k_y) \chi_{qr}^*(k_x,k_y) dk_x dk_y$$

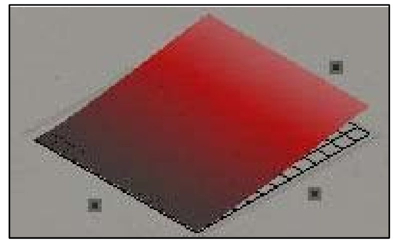
$$\chi_{mn}(k_x,k_y) = \iint_S X_m(x) Y_n(y) \exp(-i(k_x x + k_y y))$$

The experimental validation

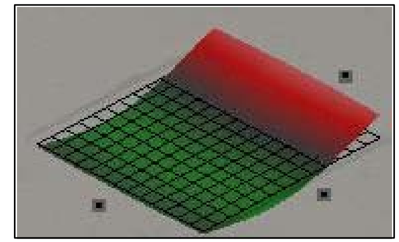




Experimental results

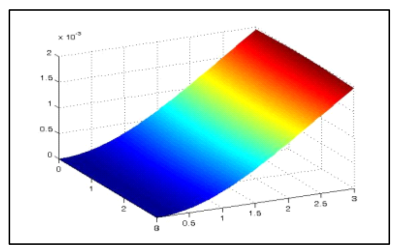


f=25kHz

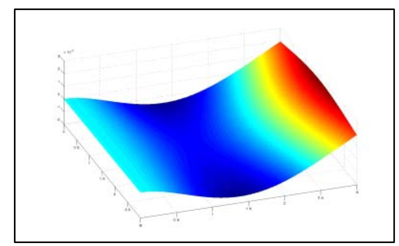


f=200kHz

Theoretical modes



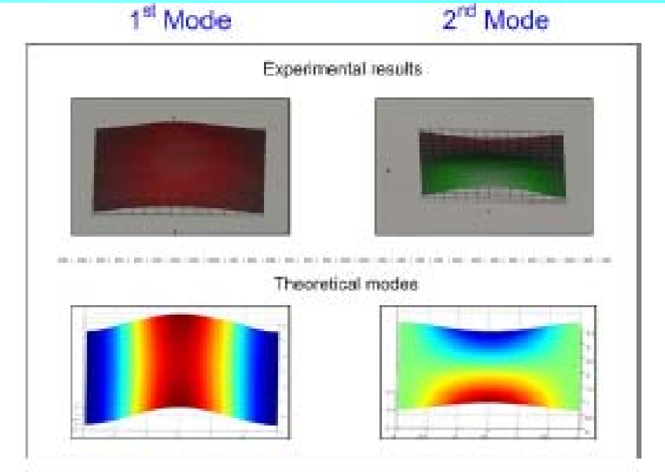
f=23kHz



f=190kHz

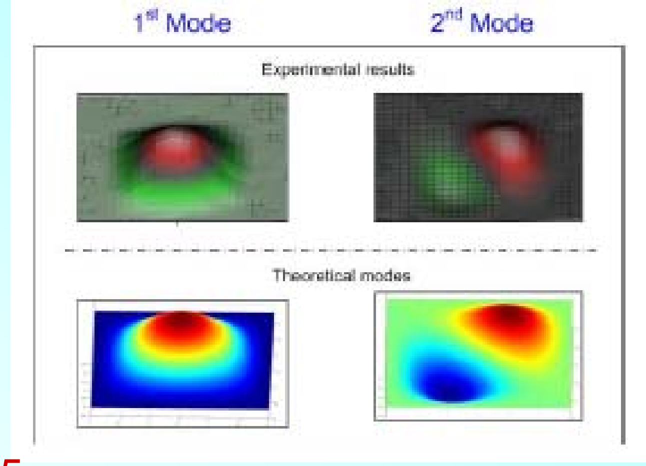
Plate dimension: 294x295X5 μm, cantilever BC

Modes	In Vacuo	In Air		In Water	
		Theo.	Expe.	Theo.	Expe.
1st	436.80	433.3	371.87	130.40	117.2
2nd	537.59	534.5	542.19	227.98	278.1
3rd	911.98	909.6	929.06	453.65	405.0

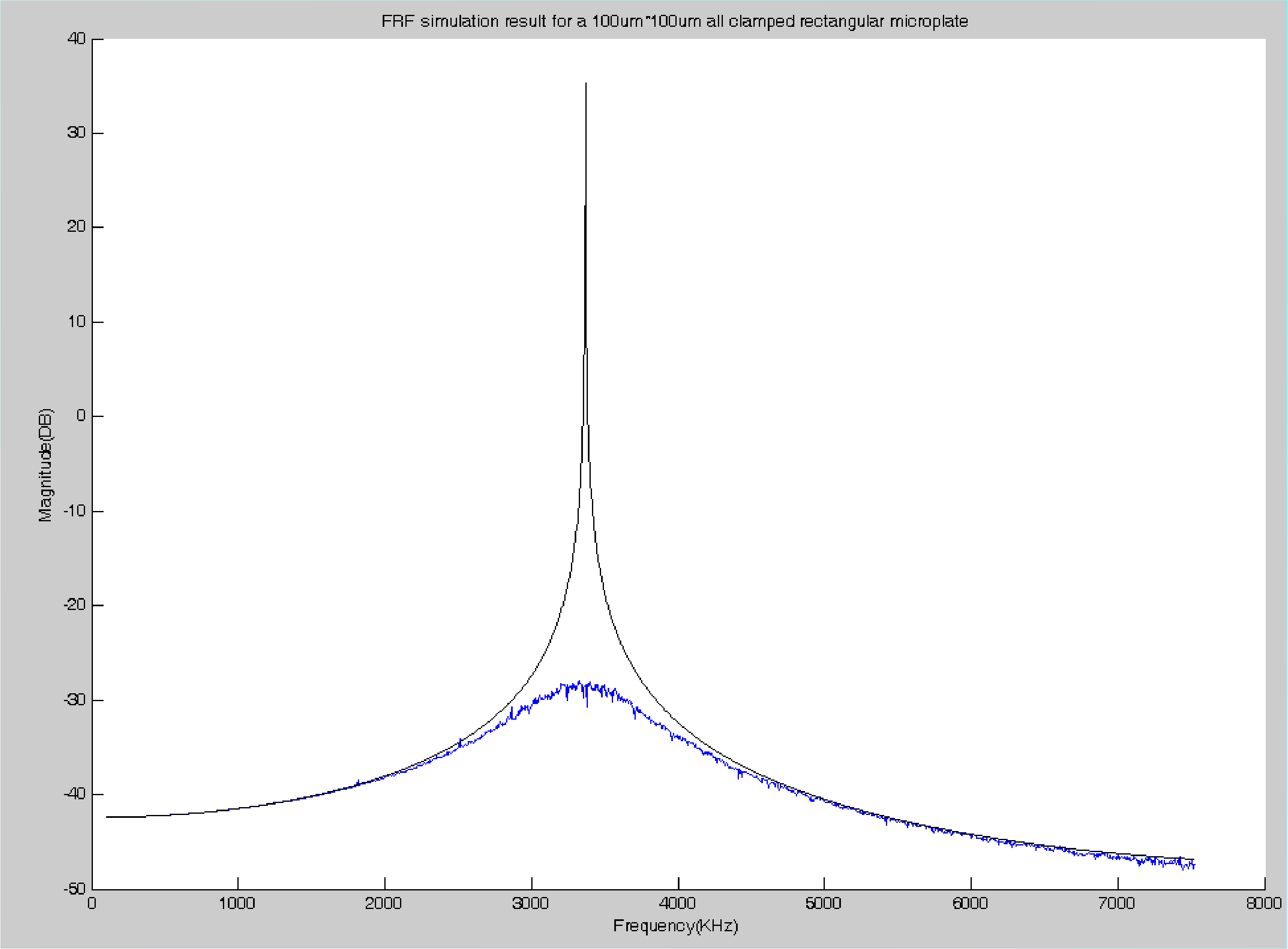


Two opposite edges clamped and the other two edge free, 296μm*309 μm*5 μm

Modes	In Vacuo	In Air		In Water	
		Theo.	Expe.	Theo.	Expe.
1st	523.08	519.67	460.0	152.67	153.44
2nd	1065.8	1062.6	1018.1	436.44	445.3
4th	1581.3	1577.8	1542.8	733.78	793.75



All edges clamped, 358μm*360 μm*5 μm



The comparison of the simplified solution and the realistic model solution

Conclusion

- A 3-D theoretical model and its analytical solution for the vibration of microplates in compressible viscous fluid is developed.
- The hydrodynamic force applied by fluid is developed.
- Fluid-loading impedance matrix is derived and used to study the effect of acoustic radiations and viscous loss.
- Experimental results correspond well with analytical simulations.

Thoughts on joints?